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A class of exactly-solvable eigenvalue problems

Carl M Bender and Qinghai Wang

Department of Physics, Washington University, St. Louis, MO 63130, USA

E-mail: cmb@howdy.wustl.edu and qwang@hbar.wustl.edu

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Abstract

The class of differential equation eigenvalue problems $-y''(x) + x^{2N+2}y(x) = x^N Ey(x)$ $(N = -1, 0, 1, 2, 3, \ldots)$ on the interval $-\infty < x < \infty$ can be solved in closed form for all the eigenvalues E and the corresponding eigenfunctions $y(x)$. The eigenvalues are all integers and the eigenfunctions are all confluent hypergeometric functions. The eigenfunctions can be rewritten as products of polynomials and functions that decay exponentially as $x \to \pm\infty$. For odd N the polynomials that are obtained in this way are new and interesting classes of orthogonal polynomials. For example, when $N = 1$, the eigenfunctions are orthogonal polynomials in x^3 multiplying Airy functions of x^2. The properties of the polynomials for all N are described in detail.

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1. Introduction

In this paper we consider the class of differential equation eigenvalue problems

$$-y''(x) + x^{2N+2}y(x) = x^N Ey(x) \qquad (N = -1, 0, 1, 2, 3, \ldots) \tag{1.1}$$

on the interval $-\infty < x < \infty$. The eigenfunction $y(x)$ is required to obey the boundary conditions that $y(x)$ vanish exponentially rapidly as $x \to \pm\infty$. For each integer $N \geqslant -1$, it is possible to solve these eigenvalue problems in closed form. The eigenvalues are all integers and the associated eigenfunctions are all confluent hypergeometric functions. Furthermore, all the eigenfunctions for each value of N can be written as the product of a polynomial and a given function that vanishes exponentially for large $|x|$. The classes of polynomials that are obtained in this way are orthogonal and for odd N are apparently new and have interesting mathematical properties.

The eigenvalue problem (1.1) discussed in this paper arises in many contexts. In classical physics a perturbative technique called boundary layer theory has been developed to find approximate solutions to boundary value problems of the form

$$\epsilon w''(x) + a(x)w'(x) + b(x)w(x) = 0 \qquad w(-1) = A, w(1) = B \tag{1.2}$$

where ϵ is treated as a small parameter. Problems of this sort appear in the study of fluid-flow problems in various geometries. Perturbative treatments of this equation are usually quite straightforward [1]. However, there is a particularly difficult special case of (1.2) that may occur when there is a *resonant* internal boundary layer. Suppose that $a(0) = 0$, so that there is an internal boundary layer at $x = 0$. Suppose further that near $x = 0, a(x) \sim \alpha x$ and $b(x) \sim \beta$. Then, near $x = 0$ the differential equation in (1.2) is approximated by

$$\epsilon w''(x) + \alpha x w'(x) + \beta w(x) = 0. \tag{1.3}$$

The Gaussian change of variables $w(x) = \mathrm{e}^{-\alpha x^2/(4\epsilon)} y(x)$ converts (1.3) to Schrödinger form:

$$-y''(x) + \frac{\alpha^2}{4\epsilon^2} x^2 y(x) = \frac{2\beta - \alpha}{2\epsilon} y(x). \tag{1.4}$$

Apart from scaling, this equation is the $N = 0$ case of (1.1). It describes the quantum harmonic oscillator, and eigenvalues occur when the parameters α and β satisfy

$$\beta = (n+1)\alpha \qquad (n = 0, 1, 2, 3, \ldots). \tag{1.5}$$

When the parameters α and β are related in this fashion, the internal boundary layer is said to be *resonant*. Unlike conventional boundary layers, a resonant boundary layer is not narrow; that is, its thickness is not small when ϵ is small. As a result it is particularly difficult to treat the resonant case using ordinary boundary layer methods [1].

More generally, if $a(x) \sim \alpha x^{N+1}$ and $b(x) \sim \beta x^N$ when x is near 0, we obtain the differential equation in (2.2). This equation is then converted to the differential equation in (1.1) by the exponential change of variables in (2.1). Thus, the eigenvalue problem in (1.1) characterizes the *general* resonant case in the theory of internal boundary layers. The special case of the harmonic oscillator discussed above occurs when $N = 0$.

The eigenvalue problem also arises in the context of supersymmetric quantum mechanics and quasi-exactly solvable models. It appears, for example, in the recent study of Voros [2] and Dorey *et al* [3]. The same differential equation was also considered by Znojil [4] but with boundary conditions imposed on a semi-infinite interval. The quantum problem in (1.1) may be thought of as a peculiar inverse approach to quasi-exact solvability. Ordinarily, in this field one tries to construct potentials for which a finite number of eigenvalues of the spectrum can be calculated exactly and in closed form, while the remaining part of the spectrum remains analytically intractable. The problem in (1.1) is to construct potentials $V(x)$ for which there is an eigenvalue that is exactly zero. The zero eigenvalue may or may not be the ground-state energy of the potential $V(x) = x^{2N+2} - x^N E$ that has been constructed. In this paper we will see that the case of odd-integer N is much more interesting than the even-N case.

The eigenvalue problem in (1.1) is especially interesting because, as we show in section 2, leading-order WKB theory (physical optics) gives the exact spectrum E for all odd N and almost the exact answer for even N.

This paper is organized as follows. In section 2 we give the exact solution to the eigenvalue problem in (1.1). We show that the eigenvalues E are all integers for each value of $N = -1, 0, 1, 2, 3, \ldots$ and that the corresponding eigenfunctions are all confluent hypergeometric functions. In section 3 we examine the eigenfunctions for even-integer N. For this case the eigenspectrum is positive and the nth eigenfunction has definite parity. The nth eigenfunction has the form of a polynomial of degree n and of argument x^{N+2} multiplied by the exponential $\exp(-x^{N+2})$, which decays as $x \to \pm\infty$. The polynomials are generalized Laguerre polynomials. The polynomials for even n form an orthogonal set and the polynomials for odd n form a different orthogonal set. In section 4, we study the eigenfunctions for odd-integer N. For this case the spectrum of eigenvalues E ranges from $-\infty$ to ∞ and the nth eigenfunction does not exhibit definite parity. For each N the nth eigenfunction has the general form $xA_N(x^{N+1})P_n(x^{N+2}) + A'_N(x^{N+1})Q_n(x^{N+2})$. Here, $P_n(z)$ and $Q_n(z)$ are polynomials of degree n that satisfy the same recursion relation but have different initial conditions. The functions $A_n(z)$ are independent of n and are the *generalized Airy functions* that obey the differential equation $A''_N(z) = z^{2/(N+1)}A_N(z)$. When $N = 1$, the function $A_n(z)$ is the conventional Airy function $\mathrm{Ai}(z)$. In section 5 we consider the special cases $N = -1$ and $N = 1$ (the Airy case).

We emphasize that the eigenvalues that are obtained in this paper are not the energies of a conventional Schrödinger equation. However, the eigenfunctions that are obtained might well be useful for solving some conventional Schrödinger equations. For example, it might be useful to express the eigenfunctions of the pure anharmonic oscillator problem, $-y''(x) + x^4 y(x) = Ey(x)$ as linear combinations of the eigenfunctions found in this paper. The work done in this paper suggests that it may well be advantageous to expand the function $y(x)$ as $P(x)\mathrm{Ai}(x^2) + Q(x)\mathrm{Ai}'(x^2)$, where P and Q are series in powers of x.

2. Exact solution of the eigenvalue problem

We solve for the eigenvalues of the differential equation (1.1) by converting it to a confluent hypergeometric equation and then imposing the boundary conditions. We begin by making the substitution

$$y(x) = \mathrm{e}^{-\frac{1}{N+2}x^{N+2}} w(x). \tag{2.1}$$

The function $w(x)$ then satisfies the differential equation

$$w''(x) - 2x^{N+1}w'(x) + \beta x^N w(x) = 0 \tag{2.2}$$

where

$$\beta \equiv E - N - 1. \tag{2.3}$$

There are now two cases to consider, N even and N odd. Suppose first that N is even. There are two linearly independent solutions to the differential equation (2.2):

$$w(x) = {}_1F_1\left(-\frac{\beta}{2(N+2)}, 1 - \frac{1}{N+2}; \frac{2}{N+2}x^{N+2}\right) \tag{2.4}$$

and

$$w(x) = x\,{}_1F_1\left(\frac{1}{N+2} - \frac{\beta}{2(N+2)}, 1 + \frac{1}{N+2}; \frac{2}{N+2}x^{N+2}\right). \tag{2.5}$$

When the first parameter of the confluent hypergeometric function is a negative integer, its Taylor series truncates to a Laguerre polynomial:

$${}_1F_1(-n, c; t) = n!\,\frac{\Gamma(c)}{\Gamma(c-n)}\,\mathrm{L}_n^{(c-1)}(t). \tag{2.6}$$

Thus, for the solution in (2.4) we obtain

$$\frac{\beta_n}{2(N+2)} = n \qquad (n = 0, 1, 2, 3, \ldots) \tag{2.7}$$

or

$$E_n = 2n(N+2) + N + 1 \qquad (n = 0, 1, 2, 3, \ldots). \tag{2.8}$$

The corresponding eigenfunctions have even parity. For the solution in (2.5), we obtain

$$-\frac{1}{N+2}+\frac{\beta_n}{2(N+2)}=n \qquad (n=0,1,2,3,\ldots) \tag{2.9}$$

or

$$E_n=2n(N+2)+N+3 \qquad (n=0,1,2,3,\ldots). \tag{2.10}$$

The corresponding eigenfunctions have odd parity.

When N is odd, the solution to the differential equation (2.2) that is well behaved as $x\to\infty$, is a particular linear combination of the solutions in (2.4) and (2.5) known as a confluent hypergeometric function of the second kind:

$$\begin{aligned} w(x)=&\frac{\Gamma\left(\frac{1}{N+2}\right)}{\Gamma\left(\frac{1}{N+2}-\frac{\beta}{2(N+2)}\right)}\,{}_1F_1\left(-\frac{\beta}{2(N+2)},1-\frac{1}{N+2};\frac{2}{N+2}x^{N+2}\right)\\ &+\frac{\Gamma\left(-\frac{1}{N+2}\right)}{\Gamma\left(-\frac{\beta}{2(N+2)}\right)}x\,{}_1F_1\left(\frac{1}{N+2}-\frac{\beta}{2(N+2)},1+\frac{1}{N+2};\frac{2}{N+2}x^{N+2}\right). \end{aligned} \tag{2.11}$$

Note that as $x\to+\infty$, the function $y(x)$ in (2.1) vanishes exponentially. However, as $x\to-\infty$, the exponential factor in (2.1) grows and we have

$$\begin{aligned} y(x)\sim\mathrm{e}^{-\frac{1}{N+2}x^{N+2}}\left(-\frac{2}{N+2}x^{N+2}\right)^{\frac{\beta}{2(N+2)}}&\left[\frac{\Gamma\left(\frac{1}{N+2}\right)\Gamma\left(1-\frac{1}{N+2}\right)}{\Gamma\left(\frac{1}{N+2}-\frac{\beta}{2(N+2)}\right)\Gamma\left(1-\frac{1}{N+2}+\frac{\beta}{2(N+2)}\right)}\right.\\ &\left.-\frac{\Gamma\left(-\frac{1}{N+2}\right)\Gamma\left(1+\frac{1}{N+2}\right)}{\Gamma\left(-\frac{\beta}{2(N+2)}\right)\Gamma\left(1+\frac{\beta}{2(N+2)}\right)}\right]\left[1+\mathrm{O}\left(\frac{1}{|x|^{N+2}}\right)\right]. \end{aligned} \tag{2.12}$$

Because $\exp\left(-\frac{1}{N+2}x^{N+2}\right)$ grows exponentially as $x\to-\infty$, the only way to satisfy the boundary condition as $x\to-\infty$ is for the expression in square brackets to vanish. The expression in square brackets simplifies to

$$\frac{\sin\left[\left(\frac{1}{N+2}-\frac{\beta}{2(N+2)}\right)\pi\right]-\sin\frac{\beta\pi}{2(N+2)}}{\sin\left(\frac{\pi}{N+2}\right)}. \tag{2.13}$$

Hence,

$$\frac{\beta_n}{N+2}=\frac{1}{N+2}+2n \qquad (n=0,\pm1,\pm2,\pm3,\ldots) \tag{2.14}$$

and thus we obtain the eigenvalues

$$E_n=(2n+1)(N+2) \qquad (n=0,\pm1,\pm2,\pm3,\ldots). \tag{2.15}$$

It is interesting that a leading-order WKB analysis (the physical optics approximation) of (1.1) gives the exact eigenvalues for odd N and almost the exact eigenvalues when N is even. Consider the following two-turning-point time-independent Schrödinger equation boundary value problem

$$-y''(x)+Q(x)y(x)=0 \qquad y(\pm\infty)=0. \tag{2.16}$$

Ordinarily, $Q(x)=V(x)-E$, where $V(x)$ is the potential and E is the energy. In the physical optics approximation, the condition for a solution to this problem to exist is

$$\int_A^B \mathrm{d}x\sqrt{-Q(x)}=\left(n+\frac{1}{2}\right)\pi \qquad (n=0,1,2,3,\ldots) \tag{2.17}$$

where the turning points A and B satisfy $Q(A) = Q(B) = 0$. If we apply the quantization condition (2.17) to (1.1), where $Q(x) = x^{2N+2} - Ex^N$, we obtain

$$\int_0^B \mathrm{d}x\, \sqrt{Ex^N - x^{2N+2}} = \left(n + \frac{1}{2}\right)\pi \qquad (n = 0, 1, 2, 3, \ldots) \tag{2.18}$$

where we assume, without loss of generality, that E is positive. The turning point B satisfies $B^{N+2} = E$. This integral can be evaluated exactly as a beta function and we obtain

$$E = (2n + 1)(N + 2). \tag{2.19}$$

This is precisely the result in (2.15) for odd N. Also, it is nearly the result for even N in (2.8) and (2.10), which can be combined to read

$$E = (2n + 1)(N + 2) \pm 1. \tag{2.20}$$

A striking property of the eigenvalue problem (1.1) is that for even N the eigenvalues are positive but for odd N the eigenvalues are both positive and negative. This is reminiscent of the difference between the Klein–Gordon equation for bosons, which has positive-energy states only, and the Dirac equation for fermions, which has positive-energy states (electrons) and negative-energy states (positrons or holes).

In the next two sections we describe the two cases odd N and even N in greater depth. We consider the simpler case of even N in section 3 and turn to the more interesting case of odd N in sections 4 and 5.

3. Eigenvalue problem for even N

When N is even, all the eigenvalues are positive and the eigenfunctions have either even or odd parity. The even-parity eigenfunctions have the form

$$y_{2n}(x) = \mathrm{e}^{-\frac{1}{N+2}x^{N+2}} \mathrm{L}_n^{(-\frac{1}{N+2})}\left(\frac{2}{N+2}x^{N+2}\right) \qquad (n = 0, 1, 2, 3, \ldots) \tag{3.1}$$

and the corresponding eigenvalues are

$$E_{2n} = 2n(N + 2) + N + 1 \tag{3.2}$$

where $\mathrm{L}_n^{(\alpha)}$ is the generalized Laguerre polynomial. The odd-parity eigenfunctions are

$$y_{2n+1}(x) = \mathrm{e}^{-\frac{1}{N+2}x^{N+2}} x\, \mathrm{L}_n^{(\frac{1}{N+2})}\left(\frac{2}{N+2}x^{N+2}\right) \qquad (n = 0, 1, 2, 3, \ldots) \tag{3.3}$$

and the corresponding eigenvalues are

$$E_{2n+1} = 2n(N + 2) + N + 3. \tag{3.4}$$

Note that the eigenfunctions have the form of a decaying exponential multiplying a polynomial. For the even-parity solutions we can write the polynomial as a monic[1] polynomial $p_n(z)$ in the variable $z = 4x^{N+2}$:

$$p_n(z) = (-1)^n n!\,[2(N + 2)]^n\, \mathrm{L}_n^{(-\frac{1}{N+2})}\left(\frac{z}{2(N + 2)}\right) \tag{3.5}$$

and for the odd-parity solutions, we have

$$q_n(z) = (-1)^n n!\,[2(N + 2)]^n\, \mathrm{L}_n^{(\frac{1}{N+2})}\left(\frac{z}{2(N + 2)}\right). \tag{3.6}$$

[1] The term *monic* means that the coefficient of the highest power in the polynomial is 1.

These polynomials satisfy very similar recurrence relations

$$\begin{aligned} p_{n+1}(z) &= [z-2(N+2)(2n+1)+2]p_n(z)-2(N+2)n[2(N+2)n-2]p_{n-1}(z)\\ q_{n+1}(z) &= [z-2(N+2)(2n+1)-2]q_n(z)-2(N+2)n[2(N+2)n+2]q_{n-1}(z) \end{aligned} \tag{3.7}$$

where the initial conditions are

$$\begin{aligned} p_0(z)&=1 \qquad p_1(z)=z-2N-2\\ q_0(z)&=1 \qquad q_1(z)=z-2N-6. \end{aligned} \tag{3.8}$$

The polynomials $p_n(z)$ and $q_n(z)$ also obey similar differential equations

$$\begin{aligned} 2(N+2)zp_n''(z)+(2N+2-z)p_n'(z)+np_n(z)&=0\\ 2(N+2)zq_n''(z)+(2N+6-z)q_n'(z)+nq_n(z)&=0 \end{aligned} \tag{3.9}$$

and differential relations

$$\begin{aligned} 2(N+2)zp_n'(z)+p_{n+1}(z)+[2(N+2)(n+1)-2-z]p_n(z)&=0\\ 2(N+2)zq_n'(z)+q_{n+1}(z)+[2(N+2)(n+1)+2-z]q_n(z)&=0. \end{aligned} \tag{3.10}$$

The generating functions for these polynomials are also quite similar:

$$\begin{aligned} G_p(z,t) &\equiv \sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}p_n(z)=[1-2(N+2)t]^{-1+\frac{1}{N+2}}\,\mathrm{e}^{\frac{zt}{2(N+2)t-1}}\\ G_q(z,t) &\equiv \sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}q_n(z)=[1-2(N+2)t]^{-1-\frac{1}{N+2}}\,\mathrm{e}^{\frac{zt}{2(N+2)t-1}}. \end{aligned} \tag{3.11}$$

The polynomials $p_n(z)$ and $q_n(z)$ are separately orthogonal:

$$\begin{aligned} \int_0^\infty \mathrm{d}z\, w_p(z)p_m(z)p_n(z) &= [2(N+2)]^{2n}n!\,\frac{\Gamma\left(n+1-\frac{1}{N+2}\right)}{\Gamma\left(1-\frac{1}{N+2}\right)}\,\delta_{m,n}\\ \int_0^\infty \mathrm{d}z\, w_q(z)q_m(z)q_n(z) &= [2(N+2)]^{2n}n!\,\frac{\Gamma\left(n+1+\frac{1}{N+2}\right)}{\Gamma\left(1+\frac{1}{N+2}\right)}\,\delta_{m,n} \end{aligned} \tag{3.12}$$

where the weight functions $w_p(z)$ and $w_q(z)$ are given by

$$\begin{aligned} w_p(z) &= \mathrm{e}^{-\frac{z}{2(N+2)}}z^{-\frac{1}{N+2}}[2(N+2)]^{\frac{1}{N+2}-1}\frac{1}{\Gamma\left(1-\frac{1}{N+2}\right)}\\ w_q(z) &= \mathrm{e}^{-\frac{z}{2(N+2)}}z^{\frac{1}{N+2}}[2(N+2)]^{-\frac{1}{N+2}-1}\frac{1}{\Gamma\left(1+\frac{1}{N+2}\right)}. \end{aligned} \tag{3.13}$$

The moments of these weight functions are

$$\begin{aligned} a_n^{(p)} &\equiv \int_0^\infty \mathrm{d}z\, w_p(z)z^n = [2(N+2)]^n\frac{\Gamma\left(n+1-\frac{1}{N+2}\right)}{\Gamma\left(1-\frac{1}{N+2}\right)}\\ a_n^{(q)} &\equiv \int_0^\infty \mathrm{d}z\, w_q(z)z^n = [2(N+2)]^n\frac{\Gamma\left(n+1+\frac{1}{N+2}\right)}{\Gamma\left(1+\frac{1}{N+2}\right)}. \end{aligned} \tag{3.14}$$

The (divergent) power series constructed from these moments have particularly simple continued-fraction expansions in which the continued-fraction coefficients are all integers:

$$\sum_{n=0}^{\infty} a_n^{(p)} t^n = \cfrac{1}{1-\cfrac{[2(N+2)-2]t}{1-\cfrac{[2(N+2)]t}{1-\cfrac{[4(N+2)-2]t}{1-\cfrac{[4(N+2)]t}{1-\cfrac{[6(N+2)-2]t}{1-\cfrac{[6(N+2)]t}{1-\cdots}}}}}}}$$

$$\sum_{n=0}^{\infty} a_n^{(q)} t^n = \cfrac{1}{1-\cfrac{[2(N+2)+2]t}{1-\cfrac{[2(N+2)]t}{1-\cfrac{[4(N+2)+2]t}{1-\cfrac{[4(N+2)]t}{1-\cfrac{[6(N+2)+2]t}{1-\cfrac{[6(N+2)]t}{1-\cdots}}}}}}}. \tag{3.15}$$

We illustrate these general results for the two special cases $N = 0$ and $N = 2$.

Special case $N = 0$: The Harmonic oscillator. For this case the eigenvalues in (3.2) are $E = 1, 3, 5, 7, \ldots$ and the polynomials $p_n(z)$ and $xq_n(z)$ in (3.5) and (3.6) coalesce to become the standard Hermite polynomials $\mathrm{H}_n(x)$:

$$\begin{aligned} p_n(z) &= \mathrm{H}_{2n}(x) = (-1)^n 2^{2n} n!\, \mathrm{L}_n^{(-1/2)}(x^2) \\ xq_n(z) &= \mathrm{H}_{2n+1}(x) = (-1)^n 2^{2n+1} n!\, x\mathrm{L}_n^{(1/2)}(x^2). \end{aligned} \tag{3.16}$$

Special case $N = 2$. For this case the eigenvalues in (3.2) are $E = 3, 5, 11, 13, 19, 21, \ldots$ and the first few monic polynomials $p_n(x)$ and $q_n(x)$ in (3.5) and (3.6) are

$$\begin{aligned} &p_0(z) = 1 \qquad p_1(z) = z - 6 \qquad p_2(z) = z^2 - 28z + 84 \\ &p_3(z) = z^3 - 66z^2 + 924z - 1848 \\ &p_4(z) = z^4 - 120z^3 + 3960z^2 - 36960z + 55440 \\ &q_0(z) = 1 \qquad q_1(z) = z - 10 \qquad q_2(z) = z^2 - 36z + 180 \\ &q_3(z) = z^3 - 78z^2 + 1404z - 4680 \\ &q_4(z) = z^4 - 136z^3 + 5304z^2 - 63648z + 159120. \end{aligned} \tag{3.17}$$

4. Eigenvalue problem for odd N

For this case the eigenvalues are

$$E_n = (2n+1)(N+2) \qquad (n = 0, \pm 1, \pm 2, \pm 3, \ldots) \tag{4.1}$$

and the corresponding eigenfunctions $y_n(x)$, which do not have definite parity, are confluent hypergeometric functions of the second kind:

$$y_n(x) = \mathrm{e}^{-\frac{1}{N+2}x^{N+2}}\, \mathrm{U}\left(-n - \frac{1}{2(N+2)}, 1 - \frac{1}{N+2}; \frac{2}{N+2}x^{N+2}\right). \tag{4.2}$$

Note that the boundary condition as $x \to +\infty$ is already satisfied and the quantization comes from requiring that $y(x) \to 0$ as $x \to -\infty$.

For each N the eigenfunctions $y_n(x)$ in (4.2) can be expressed in terms of what we will call *generalized Airy functions* $A_N(x)$ combined with polynomials P_n and Q_n as follows:[2]

$$y_{-n-1}(x) = 2^{-\frac{N+1}{2(N+2)}} x A_N\left[\left(2^{-\frac{N+1}{2(N+2)}} x\right)^{N+1}\right] P_n(4x^{N+2}) + A'_N\left[\left(2^{-\frac{N+1}{2(N+2)}} x\right)^{N+1}\right] Q_n(4x^{N+2}) \qquad (n \geqslant 0) \tag{4.3}$$

and

$$y_n(x) = y_{-n-1}(-x) \qquad (n \geqslant 0). \tag{4.4}$$

We define the *generalized Airy function* $A_N(x)$ as the solution to the differential equation

$$A''_N(x) = x^{\frac{2}{N+1}} A_N(x) \tag{4.5}$$

that decays exponentially as $x \to +\infty$. Note that when $N = 1$, the function $A_1(x)$ is just the conventional Airy function $\mathrm{Ai}(x)$. We can express $A_N(x)$ in terms of the associated Bessel function $K_\nu(z)$ as follows:

$$A_N(x) = \frac{1}{2\pi}[4(N+1)]^{\frac{N+1}{2(N+2)}} \sqrt{\frac{x}{N+2}} K_{\frac{N+1}{2(N+2)}}\left(\frac{N+1}{N+2} x^{\frac{N+2}{N+1}}\right). \tag{4.6}$$

With this choice the function $A_N(x)$ is normalized so that

$$\int_0^\infty \mathrm{d}x\, A_N(x) = \frac{1}{\pi}\Gamma\left(\frac{N+1}{N+2}\right)\Gamma\left(\frac{N+1}{2(N+2)}\right) 2^{-\frac{N+7}{2(N+2)}} (N+1)^{\frac{1}{N+2}} (N+2)^{-\frac{3}{2(N+2)}}. \tag{4.7}$$

Note that this reduces to the standard result $\int_0^\infty \mathrm{d}x\, \mathrm{Ai}(x) = \frac{1}{3}$ when $N = 1$.

The polynomials $P_n(z)$ and $Q_n(z)$, where $z = 4x^{N+2}$, both satisfy the *same* recursion relation

$$\begin{aligned} P_{n+1}(z) &= [z + 2(N+2)(2n+1)]P_n(z) - [2(N+2)n - 1][2(N+2)n + 1]P_{n-1}(z) \\ Q_{n+1}(z) &= [z + 2(N+2)(2n+1)]Q_n(z) - [2(N+2)n - 1][2(N+2)n + 1]Q_{n-1}(z) \end{aligned} \tag{4.8}$$

but have different initial conditions

$$\begin{aligned} P_0(z) &= 1 \qquad P_1(z) = z + 2N + 5 \\ Q_0(z) &= 1 \qquad Q_1(z) = z + 2N + 3. \end{aligned} \tag{4.9}$$

The polynomials $P_n(z)$ and $Q_n(z)$ satisfy *coupled* second-order differential equations

$$\begin{aligned} 4(N+2)zP''_n(z) + 4(N+3)P'_n(z) + 2zQ'_n(z) + Q_n(z) &= (2n+1)P_n(z) \\ 4(N+2)zQ''_n(z) + 4(N+1)Q'_n(z) + 2zP'_n(z) + P_n(z) &= (2n+1)Q_n(z) \end{aligned} \tag{4.10}$$

and *coupled* differential relations

$$\begin{aligned} 4(N+2)zP'_n(z) &= 2P_{n+1}(z) - zQ_n(z) - [4(N+2)(n+1) + 2 + z]P_n(z) \\ 4(N+2)zQ'_n(z) &= 2Q_{n+1}(z) - zP_n(z) - [4(N+2)(n+1) - 2 + z]Q_n(z). \end{aligned} \tag{4.11}$$

[2] To avoid confusion, for odd N we use upper-case notation P_n and Q_n to represent the polynomials; we use lower-case notation p_n and q_n to represent the polynomials associated with even N.

The generating functions for these polynomials are

$$G_P(z,t) \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(z) = [1-2(N+2)t]^{-1-\frac{1}{2(N+2)}} \frac{{}_1F_1\left(1+\frac{1}{2(N+2)}, 1+\frac{1}{N+2}; \frac{z}{2(N+2)[1-2(N+2)t]}\right)}{{}_1F_1\left(1+\frac{1}{2(N+2)}, 1+\frac{1}{N+2}; \frac{z}{2(N+2)}\right)} \tag{4.12}$$

and

$$G_Q(z,t) \equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} Q_n(z) = [1-2(N+2)t]^{-1+\frac{1}{2(N+2)}} \frac{{}_1F_1\left(1-\frac{1}{2(N+2)}, 1-\frac{1}{N+2}; \frac{z}{2(N+2)[1-2(N+2)t]}\right)}{{}_1F_1\left(1-\frac{1}{2(N+2)}, 1-\frac{1}{N+2}; \frac{z}{2(N+2)}\right)}. \tag{4.13}$$

The polynomials $P_n(z)$ and $Q_n(z)$ obey identical-looking orthogonality and normalization conditions

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{d}x\, W_P(x) P_m(x) P_n(x) &= \frac{1}{\pi} \sin\frac{\pi}{2(N+2)} [2(N+2)]^{2n+1} \Gamma \\ &\quad\times \left(n+1-\frac{1}{2(N+2)}\right) \Gamma\left(n+1+\frac{1}{2(N+2)}\right) \delta_{m,n} \\ \int_{-\infty}^{\infty} \mathrm{d}x\, W_Q(x) Q_m(x) Q_n(x) &= \frac{1}{\pi} \sin\frac{\pi}{2(N+2)} [2(N+2)]^{2n+1} \Gamma\left(n+1-\frac{1}{2(N+2)}\right) \\ &\quad\times \Gamma\left(n+1+\frac{1}{2(N+2)}\right) \delta_{m,n}.\end{aligned} \tag{4.14}$$

The weight functions $W_P(x)$ and $W_Q(x)$ are real and positive and are expressible as principal-part integrals:

$$\begin{aligned} W_P(x) &= \int_{-\infty}^{x} \mathrm{d}s\, \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-s} \ln\left[\sqrt{\frac{-t}{2(N+2)\pi}}\, \mathrm{e}^{-\frac{t}{4(N+2)}}\, K_{\frac{N+3}{2(N+2)}}\left(\frac{-t}{4(N+2)}\right)\right] \\ W_Q(x) &= \int_{-\infty}^{x} \mathrm{d}s\, \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-s} \ln\left[\sqrt{\frac{-t}{2(N+2)\pi}}\, \mathrm{e}^{-\frac{t}{4(N+2)}}\, K_{\frac{N+1}{2(N+2)}}\left(\frac{-t}{4(N+2)}\right)\right] \end{aligned} \tag{4.15}$$

or, in terms of the generalized Airy functions $A_N(x)$,

$$\begin{aligned} W_P(x) &= \int_{-\infty}^{x} \mathrm{d}s\, \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-s} \ln\left\{-\sqrt{2\pi}(-t)^{\frac{1}{2(N+2)}} \mathrm{e}^{-\frac{t}{4(N+2)}} A_N'\left[\left(\frac{-t}{4(N+1)}\right)^{\frac{N+1}{N+2}}\right]\right\} \\ W_Q(x) &= \int_{-\infty}^{x} \mathrm{d}s\, \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-s} \ln\left\{\sqrt{2\pi}(-t)^{\frac{1}{2(N+2)}} \mathrm{e}^{-\frac{t}{4(N+2)}} A_N\left[\left(\frac{-t}{4(N+1)}\right)^{\frac{N+1}{N+2}}\right]\right\} \end{aligned} \tag{4.16}$$

where $\mathcal{P}$ indicates principal-part integration and the integral is performed on the sheet for which $-1 \equiv \mathrm{e}^{-i\pi}$.

The moments of the weight functions $W_P(x)$ and $W_Q(x)$ are given by

$$a_n^{(P)} \equiv \int_{-\infty}^{\infty} \mathrm{d}x\, x^n W_P(x) \text{ and } a_n^{(Q)} \equiv \int_{-\infty}^{\infty} \mathrm{d}x\, x^n W_Q(x). \tag{4.17}$$

The divergent power series constructed from these moments have remarkably simple continued-fraction expansions in which the continued-fraction coefficients are all integers:

$$\sum_{n=0}^{\infty} a_n^{(P)} t^n = \cfrac{1}{1-\cfrac{[2(N+2)+1]t}{1-\cfrac{[2(N+2)-1]t}{1-\cfrac{[4(N+2)+1]t}{1-\cfrac{[4(N+2)-1]t}{1-\cfrac{[6(N+2)+1]t}{1-\cfrac{[6(N+2)-1]t}{1-\cdots}}}}}}}$$

$$\sum_{n=0}^{\infty} a_n^{(Q)} t^n = \cfrac{1}{1-\cfrac{[2(N+2)-1]t}{1-\cfrac{[2(N+2)+1]t}{1-\cfrac{[4(N+2)-1]t}{1-\cfrac{[4(N+2)+1]t}{1-\cfrac{[6(N+2)-1]t}{1-\cfrac{[6(N+2)+1]t}{1-\cdots}}}}}}}. \quad (4.18)$$

5. Two special cases of the odd-N eigenvalue problem

In this section we consider two interesting special cases of the odd-N eigenvalue problem; namely, $N = -1$ and $N = 1$.

Special case $N = -1$. For this case equation (4.5) is of course not valid. However, the formula for the eigenvalues E in (4.1) is still valid and $E = \pm 1, \pm 3, \pm 5, \pm 7, \ldots$. The eigenfunctions $y_n(x)$ in (4.2) are now *Bateman functions* $k_{E_n}(x)$:

$$y_n(x) = k_{E_n}(x) \equiv \frac{2}{\pi}\int_0^{\pi/2} \mathrm{d}\theta \cos(x\tan\theta - E_n\theta). \quad (5.1)$$

Apart from an overall multiplicative constant, the solution can be written in terms of associated Bessel functions combined with polynomials. For negative eigenvalues

$$y_{-n-1}(x) = xK_0(x)P_n(4x) + xK_0'(x)Q_n(4x) \qquad (n \geqslant 0) \quad (5.2)$$

and for positive eigenvalues

$$y_n(x) \equiv y_{-n-1}(-x) \qquad (n \geqslant 0). \quad (5.3)$$

Note that the eigenfunctions are finite at the origin but that there is a branch cut. For definiteness, we take the branch cut to run up the positive imaginary-x axis.

In terms of the variable $z = 4x$, the first few polynomials $P_n(z)$ and $Q_n(z)$ are

$$\begin{aligned}
&P_0(z) = 1 \qquad P_1(z) = z+3 \qquad P_2(z) = z^2+9z+15\\
&P_3(z) = z^3+19z^2+90z+105\\
&P_4(z) = z^4+33z^3+321z^2+1050z+945\\
&P_5(z) = z^5+51z^4+852z^3+5631z^2+14175z+10395\\
&Q_0(z) = 1 \qquad Q_1(z) = z+1 \qquad Q_2(z) = z^2+7z+3\\
&Q_3(z) = z^3+17z^2+58z+15 \qquad Q_4(z) = z^4+31z^3+261z^2+582z+105\\
&Q_5(z) = z^5+49z^4+756z^3+4209z^2+6927z+945.
\end{aligned} \quad (5.4)$$

The polynomials $P_n(x)$ and $Q_n(x)$ satisfy the recursion relations in (4.8) with $N = -1$, the coupled second-order differential equations in (4.10) with $N = -1$ and the coupled differential relations in (4.11) with $N = -1$. The generating functions for $P_n(z)$ and $Q_n(z)$ are expressed in terms of Bateman functions k_ν:

$$\begin{aligned} G_P(z,t) &\equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(z) = (1-2t)^{-3/2} \frac{\mathrm{e}^{\frac{z}{4(1-2t)}}\, k_{-3}\left(\frac{z}{4(1-2t)}\right)}{\mathrm{e}^{z/4}\, k_{-3}(z/4)} \\ G_Q(z,t) &\equiv \sum_{n=0}^{\infty} \frac{t^n}{n!} Q_n(z) = (1-2t)^{-1/2} \frac{\mathrm{e}^{\frac{z}{4(1-2t)}}\, k_{-1}\left(\frac{z}{4(1-2t)}\right)}{\mathrm{e}^{z/4}\, k_{-1}(z/4)}. \end{aligned} \tag{5.5}$$

The polynomials $P_n(z)$ and $Q_n(z)$ obey identical-looking orthogonality and normalization conditions

$$\int_{-\infty}^{\infty} \mathrm{d}x\, W_P(x) P_m(x) P_n(x) = \int_{-\infty}^{\infty} \mathrm{d}x\, W_Q(x) Q_m(x) Q_n(x) = (2n-1)!!\,(2n+1)!!\,\delta_{m,n} \tag{5.6}$$

where $(-1)!! = 1$. Note that the weight functions $W_P(x)$ and $W_Q(x)$ are real and positive and are expressible as principal-part integrals:

$$\begin{aligned} W_P(x) &= \int_{-\infty}^{x} \mathrm{d}s\, \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-s} \ln\left[\sqrt{\frac{-t}{2\pi}}\, \mathrm{e}^{-t/4}\, K_1(-t/4)\right] \\ W_Q(x) &= \int_{-\infty}^{x} \mathrm{d}s\, \mathcal{P} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{t-s} \ln\left[\sqrt{\frac{-t}{2\pi}}\, \mathrm{e}^{-t/4}\, K_0(-t/4)\right]. \end{aligned} \tag{5.7}$$

The moments of the weight functions $W_P(x)$ and $W_Q(x)$ give rise to the following lovely continued-fraction expansions:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n^{(P)} t^n &= \cfrac{1}{1-\cfrac{3t}{1-\cfrac{t}{1-\cfrac{5t}{1-\cfrac{3t}{1-\cfrac{7t}{1-\cfrac{5t}{1-\cdots}}}}}}} \\ \sum_{n=0}^{\infty} a_n^{(Q)} t^n &= \cfrac{1}{1-\cfrac{t}{1-\cfrac{3t}{1-\cfrac{3t}{1-\cfrac{5t}{1-\cfrac{5t}{1-\cfrac{7t}{1-\cdots}}}}}}}. \end{aligned} \tag{5.8}$$

Special case $N = 1$ (Airy functions). For this case the eigenvalues E in (4.1) are $E = \pm 3, \pm 9, \pm 15, \pm 21, \ldots$ and the eigenfunctions $y_n(x)$ in (4.2) are written in terms of Airy functions combined with polynomials. For negative eigenvalues we have

$$y_{-n-1}(x) = 2^{-1/3}\, x\, \mathrm{Ai}(2^{-2/3}x^2) P_n(4x^3) + \mathrm{Ai}'(2^{-2/3}x^2) Q_n(4x^3) \qquad (n \geqslant 0) \tag{5.9}$$

and for positive eigenvalues we have

$$y_n(x) \equiv y_{-n-1}(-x) \qquad (n \geqslant 0). \tag{5.10}$$

The polynomials P_n and Q_n are functions of the variable $z = 4x^3$. The first few such polynomials are

$$\begin{aligned}
&P_0(z) = 1 \qquad P_1(z) = z+7 \qquad P_2(z) = z^2+25z+91\\
&P_3(z) = z^3+55z^2+698z+1729\\
&P_4(z) = z^4+97z^3+2685z^2+22970z+43225 \qquad Q_0(z) = 1\\
&Q_1(z) = z+5 \qquad Q_2(z) = z^2+23z+55\\
&Q_3(z) = z^3+53z^2+602z+935\\
&Q_4(z) = z^4+95z^3+2505z^2+18790z+21505.
\end{aligned} \tag{5.11}$$

The polynomials $P_n(x)$ and $Q_n(x)$ satisfy the recursion relations (4.8), the coupled second-order differential equations (4.10) and the coupled differential relations (4.11) with $N = 1$. The generating functions $G_P(z,t)$ and $G_Q(z,t)$ and the integral representations of the weight functions W_P and W_Q are obtained by setting $N = 1$ in (4.12), (4.13), (4.15) and (4.16). The polynomials $P_n(z)$ and $Q_n(z)$ obey identical-looking orthogonality and normalization conditions

$$\int_{-\infty}^{\infty} \mathrm{d}z\, W_P(z)P_m(z)P_n(z) = \int_{-\infty}^{\infty} \mathrm{d}z\, W_Q(z)Q_m(z)Q_n(z) = \frac{3}{\pi}6^{2n}\Gamma\left(n+\frac{5}{6}\right)\Gamma\left(n+\frac{7}{6}\right)\delta_{m,n} \tag{5.12}$$

and the moments of the weight functions $\mathrm{W}_P(x)$ and $W_Q(x)$ have the following continued-fraction expansions:

$$\sum_{n=0}^{\infty} a_n^{(P)}t^n = \cfrac{1}{1-\cfrac{7t}{1-\cfrac{5t}{1-\cfrac{13t}{1-\cfrac{11t}{1-\cfrac{19t}{1-\cfrac{17t}{1-\cdots}}}}}}}$$

$$\sum_{n=0}^{\infty} a_n^{(Q)}t^n = \cfrac{1}{1-\cfrac{5t}{1-\cfrac{7t}{1-\cfrac{11t}{1-\cfrac{13t}{1-\cfrac{17t}{1-\cfrac{19t}{1-\cdots}}}}}}}. \tag{5.13}$$

There is an interesting connection between the moments $a_n^{(Q)}$ and the combinatorial numbers $C_{2n}^{[3]}$, which represent the sum of the symmetry numbers of the $2n$-vertex connected vacuum graphs in a ϕ^3 quantum field theory [5]:

$$a_n^{(Q)} = 6nC_{2n}^{[3]}4^n. \tag{5.14}$$

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References

[1] Bender C M and Orszag S A 1978 *Advanced Mathematical Methods for Scientists and Engineers* (New York: McGraw-Hill) ch 6
[2] Voros A 2000 *J. Phys. A: Math. Gen.* **33** 7423
[3] Dorey P, Dunning C and Tatao R 2001 *Preprint* hep-th/0103051
Dorey P, Dunning C and Tatao R 2001 *Preprint* hep-th/0104119
[4] Znojil M 1997 *J. Math. Phys.* **38** 5087
[5] Bender C M and Milton K A 1994 *J. Math. Phys.* **35** 364